Indirect Reciprocity Through Gossiping Can Lead to Cooperative Clusters

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***Abstract*—Explaining how cooperation can emerge, and persist over time in various species is a prime challenge for both biologists and social scientists. Whereas cooperation in non-human species might be explained through mechanisms such as kinship selection or reciprocity, this is usually regarded as insufficient to explain the extent of cooperation observed in humans. It has been theorized that indirect reciprocity—I help you, and someone else later helps me—could explain the breadth of human cooperation. Reputation is central to this idea, since it conveys important information to third parties whether to cooperate or not with a focal player. In this paper we analyze a model for reputation dynamics through gossiping, and pay specific attention to the possible cooperation networks that may arise. In this paper we show that gossiping agents can organize into cooperative clusters, i.e. cooperate within clusters, and defect between them, which can be regarded as socially balanced. We also deduce conditions for these gossiping cooperators to be evolutionary stable.**

I. Introduction

The scale of human cooperation is larger than seen in any other animal. Since free riders can take advantage of cooperators, explaining the evolution of cooperation is an interesting and important research topic [1]. Various mechanisms have been suggested that might explain the evolution of cooperation, such as kinship, direct and indirect reciprocity [2]. Indirect reciprocity is an important mechanism for the explanation of human cooperation [3], since human cooperation is not restricted to kinship or to repeated games. It is even suggested that some form of indirect reciprocity would constitute the biological basis of morality [4].

The evolution of cooperation is usually studied through the prisoner's dilemma [1]. In this game, both players have two options: give the other some benefit at some costs to himself, or not. *Indirect reciprocity* is based on the notion that if you aid someone now, you might be returned the favor at another time by someone else. The information whether or not someone has cooperated is conveyed as his reputation.

In this paper we will analyze a model where reputation is locally constructed through game interactions and transmission of information, i.e. gossiping. This in contrast to many approaches [5]–[10] that consider reputation to be objective—that is, the same reputation for all agents—which is arguably inaccurate, although some also assume that only some (so-called 'observing') agents update their (private) reputations, so that the reputation of an agent can diverge between different 'observers'. Another noteworthy exception is [11], that studies the problem of how to deal with lying when gossiping takes place. However, none of these studies have devoted any attention to cooperation networks (who will cooperate with whom), to which we will pay special attention here.

The reputation dynamics inherent in these type of models are also of interest from another point of view. It is known in the sociological literature that gossiping plays a significant part in the maintenance of norms [12], [13]. Whenever a group upholds a certain norm, those who transgress the norm are punished through social actions [14]. More specifically, the reputation of someone who transgresses a norm would be lowered, and gossiping seems to play an important role in that process [13]. As cooperation could also be explained by social norms [15] (although that begs the question how social norms emerge and are maintained), our model would also be of interest from that point of view.

In the next section, we will introduce our gossiping reputation model. The third section will discuss fixed points of these dynamics, showing that higher social influence can lead to less cooperation within the population of gossiping cooperators. Moreover, we show that undirected fixed point cooperation networks are socially balanced. In the fourth section, we will examine the evolutionary dynamics using replicator equations, and give evolutionary stability conditions for the gossiping cooperators. Finally we will give some conclusions and directions for further research.

II. Reputation Dynamics

We consider a population of n interacting agents. Each agent has two options: give the other some benefit $b > 0$ at some costs $c < b$ to himself, or not. When the first option is chosen, we say the agent has cooperated, and when the latter option is chosen, he has defected. When both agents cooperate, the payoff is $b - c$ for both. If one cooperates, but the other defects, the payoff for the first is $-c$ and for the latter b. If both agents defect, the payoff is 0, since neither receive nor give any benefit. In each round of interaction, everybody plays everybody, and gossips about their interaction. That is, we limit our analysis here to a complete interaction graph.

What option will be chosen (cooperate or defect) will depend on some reputation. Similar to Nowak and Sigmund [5] agents will cooperate whenever the reputation is positive, and will defect when it is negative. After each round of interaction,

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players will gossip about their interaction to their neighbors. They will, however, only tell agents with whom they have cooperated.

We denote by $R_{ij}(m) \in \mathbb{R}$ the reputation agent j has in the eyes of agent i, i.e. what agent i thinks of j, whether he should cooperate or defect to j in round $m \in \mathbb{N}$. The decision to cooperate is then denoted by $\alpha_{ij}(m) \in \{0, 1\}$, and we will consider

$$\alpha_{ij}(m) = \Theta(R_{ij}(m)), \quad (1)$$

where

$$\Theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (2)$$

where $\alpha_{ij}(m) = 1$ denotes cooperation and $\alpha_{ij}(m) = 0$ defection. That is, agents cooperate whenever the reputation is positive, and defect whenever it is negative.

The reputation is updated by considering both the individual interaction, and the gossip passed on by the other players. We consider the following dynamics of the reputation

$$\Delta R_{ij}(m) = R_{ij}(m+1) - R_{ij}(m) = (1-\lambda)\Delta I_{ij}(m) + \frac{\lambda}{n-2} \sum_{k \neq i,j} \Delta S_{ij}(k, m), \quad (3)$$

where $\Delta I_{ij}(m)$ is due to the individual interaction, and $\Delta S_{ij}(k, m)$ is due to the gossip from agent k, balanced by a social influence parameter $0 \leq \lambda \leq 1$, where higher λ indicates a higher social influence. We call $\Delta I_{ij}(m)$ the *individual part*, because in the absence of any social influence, the dynamics of an agent would be solely based on his own individual interaction, and call $\Delta S_{ij}(m)$ the *social part*.

For the *individual part* we consider a version of the Win-Stay-Lose-Shift (WSLS) strategy, also known as the Pavlov strategy [16]–[18]. Let us characterize this in terms of the reputation $R_{ij}(m)$. Whenever the outcome is favorable to i—both agents cooperate, or i defects but j cooperates, thereby giving i a benefit without having to pay the costs—it would like to keep playing that choice (cooperate or defect), while if the outcome is unfavorable—both defect, or i cooperates while j defects—agent i would like to change its choice. Hence, we will increase the reputation (with 1) whenever both agents cooperate or when both defect, and decrease (with 1) the reputation whenever one defects, and the other cooperates. This leads to the following individual change in the reputation of j in the eyes of i

$$\Delta I_{ij}(m) = (2\alpha_{ij}(m) - 1)(2\alpha_{ji}(m) - 1). \quad (4)$$

The *social part* is somewhat more elaborate, for which we consider again the reputation $R_{ij}(m)$. We assume a neighbour k will only gossip to agent i, whenever agent k cooperated with agent i, or when $\alpha_{ki}(m) = 1$. Agent k will then gossip about what j has done to k in the last round. That is, agent i will be informed of $\alpha_{jk}(m)$, or whether j has cooperated or not with agent k. However, depending on what i 'thinks' of k, he might consider such an action to be 'good' or 'bad'. A small overview of this idea is provided in Fig. 1.

What actions can be considered 'good' and what 'bad' is much debated [6], [8]–[10], but Ohtsuki and Iwasa demonstrated there is a small set of best performing strategies [9].

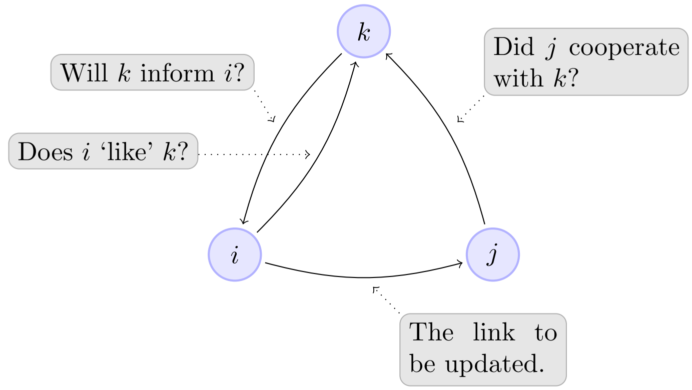

Fig. 1. Illustration of how the social part of the reputation dynamics works. In short, k informs i on the action of j only if $\alpha_{ik}(m) = 1$. Then i will consider that action as 'good' if either j cooperates and i likes k or if j defects and i doesn't like k and as 'bad' when j defects and i likes k or if j cooperates and i doesn't like k, as shown in Fig. 2.

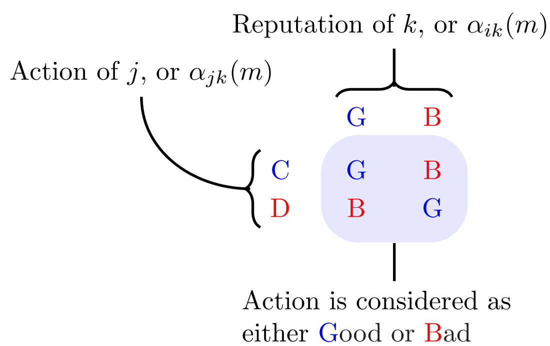

Fig. 2. The social strategy for updating the reputation $R_{ij}(m)$. If $\alpha_{jk}(m) = 1$ agent j will have cooperated, and if $\alpha_{ik}(m) = 1$ the reputation is good, and bad otherwise. Actions that are considered good increase the reputation $R_{ij}(m)$, bad ones decrease the reputation (by 1).

We will consider both cooperation with a good player, and defection with a bad player as good, and the other two options as bad, as displayed in Fig. 2, which is compatible with Ohtsuki's and Iwasa's best performing strategies. An agent k is considered good by i whenever $\alpha_{ik}(m) = 1$, and bad when $\alpha_{ik}(m) = 0$. Every good action will increase the reputation by 1 and every bad action will decrease the reputation by 1. Taking into account the fact a neighbor k will only gossip if $\alpha_{ki}(m) = 1$, we arrive at the following for the social part

$$\Delta S_{ij}(k, m) = \alpha_{ki}(m)(2\alpha_{ik}(m) - 1)(2\alpha_{jk}(m) - 1), \quad (5)$$

and take into account the average gossip of all neighbors $k \neq i, j$.

Summarizing, the reputation dynamics can be seen as a discrete non-linear switching system. If we denote by $\alpha(m)$ the $n \times n$ matrix $(\alpha_{ij}(m))$ and similarly by $R(m)$ the $n \times n$ matrix $(R_{ij}(m))$ the system can be summarized as

$$R(m+1) = R(m) + f(\alpha(m)), \quad (6)$$
$$\alpha(m) = \Theta(R(m)), \quad (7)$$

where Θ is applied elementwise, given by (2), and $f : \mathbb{B}^{n\times n} \mapsto \mathbb{R}^{n\times n}$ is given by (3), with $\mathbb{B}$ representing the Boolean domain $\mathbb{B} = \{0, 1\}$. This is the reputation dynamics

TABLE I
OVERVIEW OF VARIABLES

Reputation Dynamics	
$m \in \mathbb{N}$	Number of rounds of interaction
$R_{ij}(m) \in \mathbb{R}$	Reputation of agent j for agent i at round m
$\Delta R_{ij}(m) \in \mathbb{R}$	Change in reputation in round m
$\alpha_{ij}(m) \in \mathbb{B}$	Agent i cooperates (1) or defects (0) with agent j
$\lambda \in [0, 1]$	Social influence parameter
$p \in [0, 1]$	Probability to cooperate in the first round
Evolutionary Dynamics	
$c > 0$	Cost incurred by cooperating
$b > c$	Benefit received from cooperative partner
$x_l \in [0, 1]$	Fraction of population of type i (relative abundance)
$P_l(m) \in \mathbb{R}$	Payoff for type l after m rounds of interaction
$\bar{P}(m) \in \mathbb{R}$	Average payoff in the population

for the gossiping cooperators we will use in the remainder of the article.

III. Population structure

Let us investigate for which cooperation networks the relations of cooperation and defection endure. That is, we investigate the fixed points in terms of $\alpha_{ij}(m)$, or when $\alpha(m+1) = \alpha(m)$ in (7). That is, $\alpha(m+1) = \alpha(m) = \Theta(R(m))$, so if $\alpha_{ij}(m) = 1$, or equivalently $R_{ij}(m) \geq 0$ then $R_{ij}(m+1) \geq 0$ so that $\Delta R_{ij}(m) \geq 0$ is a sufficient condition (and likewise for $\alpha_{ij}(m) = 0$). In the following, we will call a link on which there is defection also a negative link, that is, where the reputation is negative, and likewise use positive links for cooperative links. In order for any cooperation network to be a fixed point, any positive link should remain positive, and any negative link should remain negative. We call a cooperation network a *cooperative fixed point* whenever

$$\Delta R_{ij}(m) \leq 0 \text{ if } \alpha_{ij}(m) = 0, \tag{8}$$

$$\Delta R_{ij}(m) \geq 0 \text{ if } \alpha_{ij}(m) = 1. \tag{9}$$

Using these conditions it can be proven[1] that for undirected graphs (i.e. $\alpha_{ij}(m) = \alpha_{ji}(m)$), every connected component in a cooperative fixed point network must be complete. This can be proven inductively by looking at an already completely cooperating network G. Then, if we add a new agent u, it needs to cooperate with at least one agent $v \in G$ in order to be connected. But if the agent then defects with any other agents $w \neq v \in G$, the cooperative fixed point conditions are not fulfilled, proving that, whenever two agents cooperate with each other, they should cooperate also with all their cooperating partners. This corresponds to groups, where there is only cooperation within groups, and defection between groups. Indeed, the cooperation could then be viewed as adhering to group norms, where agents are expected to cooperate with group members, and to defect with outsiders.

Let us then consider groups $1, \ldots, q$ of agents $S_1, \ldots, S_q$ with group sizes $n_1 = |S_1|, \ldots, n_q = |S_q|$, with only positive links within groups, and only negative links between groups. Now let $i, j \in S_d$, so that $\alpha_{ij}(m) = 1$. Then for this group structure to be a cooperative fixed point (9) must hold, so that

$$(1-\lambda)(2\alpha_{ij}(m)-1)(2\alpha_{ij}(m)-1)+ \frac{\lambda}{n-2}\sum_{k\neq i,j}\alpha_{ki}(m)(2\alpha_{ik}(m)-1)(2\alpha_{jk}(m)-1) \geq 0. \tag{10}$$

Since $\alpha_{ij}(m) = \alpha_{ji}(m)$ we can simplify to

$$(1-\lambda)+ \frac{\lambda}{n-2}\sum_{k\neq i,j}\alpha_{ki}(m)(2\alpha_{ik}(m)-1)(2\alpha_{jk}(m)-1) \geq 0. \tag{11}$$

Since $\alpha_{ki}(m) = 1$ only for $k \in S_d$, the summation is effectively only over members of group d. Since all links from i and j to k then are positive, we obtain

$$(1-\lambda) + \frac{\lambda}{n-2}(n_d - 2) \geq 0 \tag{12}$$

which always holds because $2 \leq n_d \leq n$ and $0 \geq \lambda \geq 1$.

Now let $i \in S_d$ and $j \in S_e$ with $e \neq d$, so that $\alpha_{ij}(m) = 0$. Similarly as before, for condition (8) to hold we require

$$(1-\lambda)- \frac{\lambda}{n-2}\sum_{k\neq i,j}\alpha_{ki}(m)(2\alpha_{ik}(m)-1)(2\alpha_{jk}(m)-1) \leq 0. \tag{13}$$

Again, since $\alpha_{ki}(m) = 1$ only for $k \in S_d$, the sum is only over agents in group d. There is always a positive link between i and $k \in S_d$ and always a negative link between j and $k \in S_d$. Hence, this simplifies to

$$(1-\lambda) - \frac{\lambda}{n-2}(n_d - 1) \leq 0, \tag{14}$$

which shows that the group size $1 \leq n_d \leq n$ is bounded by λ. More specifically, this condition needs to be fulfilled for the smallest group, thus obtaining an upper bound on the number of groups. To divide the network into q groups, the largest possible smallest group size is then n/q (namely an equipartition in q groups). Working out the previous condition then gives

$$\frac{n}{q} \geq \frac{(n-2)(1-\lambda)}{\lambda} + 1 \tag{15}$$

which reduces to approximately

$$\lambda > \frac{q}{q+1}. \tag{16}$$

Hence, the social influence λ induces an upper bound on the number of groups that can possibly exist in the network. For $\lambda < 2/3$ there can be only one group, and for increasing λ, the possible number of groups increases. For $\lambda = 1$ the number of groups is maximum, that is, there can be as many groups as there are agents; stated somewhat differently, the minimum group size is then only 1 agent.

This provides a surprising connection between indirect reciprocity and a field known in the social sciences as social balance theory [19]. When the theory was first stated a triad (a cycle of three nodes) was balanced if it contained an even number of negative links [20]. That is, if i and j have a positive relationship, they should both hold the same attitude towards

[1] We have skipped formal proofs in this paper due to lack of space.

k. A whole network can then be called balanced if all its triads are balanced. It can be proven that a network is balanced if and only if it can be divided into exactly two groups, with only positive links within groups and negative links between groups.

However, the triad with only negative links might be a little ambiguous. That is, it is not necessarily the case that "my enemy's enemy is my friend". Therefore, another definition of social balance could be provided, namely that there is not exactly one negative link in every cycle [21], which might be labeled weak balance [22]. This is equivalent to partitioning a network into multiple groups, with only positive links within and negative links between groups [20]. Various studies have investigated whether such a balanced state appears from dynamics where unbalanced triads have a probability to change into balanced triads (thereby possibly again creating other unbalanced triads) [23]–[25].

From our earlier observations, we can conclude that every fixed point cooperation graph is always weakly balanced. The inverse statement only holds for $\lambda = 1$, so the two statements are then equivalent, thus providing a new characterization of weakly balanced complete networks. So, in every weakly balanced complete network two friends will have more common friends than not, while two enemies will have less common friends than not. Stated differently, most of my friends will be friends themselves.

IV. Evolutionary Dynamics

We will now analyze how the gossiping cooperators perform against unconditional cooperators and defectors (i.e. they always cooperate or defect). That is, we let all agents play a number of rounds of the prisoner's dilemma to accumulate some payoff. Higher payoff will imply a higher reproduction rate, thereby increasing the relative abundance of successful strategies. We will focus here on the *type of agents*, not on any individual agent.

Let n_1 denote the number of unconditional cooperators, n_2 the number of unconditional defectors and $n_3 = n$ denote the number of gossiping cooperators; let the total population be denoted by $n_s = n_1 + n_2 + n_3$; let N_l for $l = 1, 2, 3$ denote respectively the set of cooperators, defectors and gossiping cooperators; and finally, let the proportion of each type be denoted by $x_l = n_l/n_s$. In the limit of large population size the evolutionary dynamics can be described by the replicator equation [26]

$$\dot{x}_l = x_l(P_l(m) - \bar{P}(m)) \tag{17}$$

where $P_l(m)$ is the payoff for type l and $\bar{P}(m) = \sum_l x_l P_l(m)$ is the average payoff in the population after m rounds. The payoff $P_l(m)$ depends in principle on the cooperative behavior of all agents, as is made explicit below, so also depends on the x_l variables. If we denote by $X = \sum_l x_l = 1$, one can see that $\dot{X} = 0$, so that the evolutionary dynamics take place on the unit simplex.

For $i \notin N_3$, that is for a cooperator or defector, we denote their action also by $\alpha_{ij}(m)$ similarly to the gossiping cooperators. However, since there are no dynamics involved for the unconditional cooperators and defectors, the $\mathbb{B}^{n_s \times n_s}$ matrix is relatively simple

$$\alpha_{ij}(m) = \begin{cases} 1 & \text{if } i \in N_1 \\ 0 & \text{if } i \in N_2 \\ \Theta(R_{ij}(m)) & \text{if } i \in N_3 \end{cases} \tag{18}$$

As the unconditional defectors and cooperators never change their action, there are no dynamics involved for them. So, only for $i \in N_3$ do we have to specify how $\alpha_{ij}(m)$, or better yet, $R_{ij}(m)$ changes, which of course remains the same as before (but only the gossiping agents are considered in the social part). Hence, for $i \in N_3$

$$R_{ij}(m+1) = R_{ij}(m) + (1-\lambda)\Delta I_{ij}(m) + \frac{\lambda}{n-2} \sum_{k \neq i, j \in N_3} \Delta S_{ij}(k, m), \tag{19}$$

and for $i \notin N_3$ nothing needs to be specified.

Consider an agent i. Each time another agent j cooperates with agent i, the first receives a benefit b and the latter pays a cost c, and similarly if agent i cooperates. So, the change in payoff for agent i can be given as $\alpha_{ji}(m)b - \alpha_{ij}(m)c$. Since we are studying the evolution in terms of *types of agents* we are interested in the average payoff for agents of a particular type. So, in general, the payoff for type l can be written as

$$P_l(m) = \sum_{w=1}^{m} \frac{1}{n_l} \sum_{i \in N_l} \frac{1}{n_s} \sum_{j \neq i} \alpha_{ji}(w)b - \alpha_{ij}(w)c. \tag{20}$$

However, this depends on the cooperative behavior of agents $\alpha_{ij}(m)$, which for gossiping cooperators can be rather difficult to deal with, depending on the initial conditions in the first round. Hence, we will only study expected values to obtain analytical results.

For the gossiping cooperators, the reputation $R_{ij}(1)$ needs to be initialized. We will set $R_{ij}(1) = 0$ and instead of setting then $\alpha_{ij}(1) = \Theta(R_{ij}(1)) = 1$, we assume gossiping agents will cooperate with probability p. So, for the first round we assume $\Pr(\alpha_{ij}(1) = 1) = p$. In order to study the evolutionary dynamics we will only look at the expected reputation $\langle R_{ij}(m) \rangle$. With increasing variance $p(1-p)$ of the probability to cooperate, we are more likely to deviate from the expected reputation, but for $p = 0$ and $p = 1$ the system is deterministic, and the derivations will be exact.

Since the unconditional defectors will always defect, and likewise the unconditional cooperators will always cooperate, the only question is what the gossiping cooperators will do. By plugging (18) into (3), and by taking into account that $\Pr(\alpha_{ij}(1) = 1) = p$ for $i \in N_3$, and only taking into account the expected change in the reputations, we arrive at the following reputations for $m = 2$

$$\langle R_{ij}(2) \rangle = \begin{cases} (2p-1)(1+\lambda(p-1)) & \text{if } j \in N_1 \\ (1-2p)(1+\lambda(p-1)) & \text{if } j \in N_2 \\ (2p-1)^2(1+\lambda(p-1)) & \text{if } j \in N_3 \end{cases} \tag{21}$$

with obviously $i \in N_3$. Then for $p > 1/2$, the expected reputation will be positive for cooperators, and negative for defectors, and for $p < 1/2$ just the other way around. If we assume that for positive expected reputation the gossiping cooperators will

TABLE II
SUMMARY OF POSSIBLE TRANSITIONS FROM A POSITIVE/NEGATIVE EXPECTED REPUTATION $\langle R_{ij}(m)\rangle$ TO A POSITIVE/NEGATIVE EXPECTED REPUTATION $\langle R_{ij}(m+1)\rangle$ IN THE NEXT ROUND VERSUS COOPERATORS AND DEFECTORS.

(a) vs. cooperators

		$\langle R_{ij}(m)\rangle$ < 0	$\langle R_{ij}(m)\rangle$ > 0
$\langle R_{ij}(m+1)\rangle$	< 0	$\lambda < \frac{1}{2}$	Never
	> 0	$\lambda > \frac{1}{2}$	Always

(b) vs. defectors

		$\langle R_{ij}(m)\rangle$ < 0	$\langle R_{ij}(m)\rangle$ > 0
$\langle R_{ij}(m+1)\rangle$	< 0	$\lambda > \frac{1}{2}$	Always
	> 0	$\lambda < \frac{1}{2}$	Never

cooperate (and for negative expected reputation defect), then the gossiping cooperators will cooperate with cooperators if $p \geq 1/2$ and defect against defectors for $p < 1/2$. Hence, this warrants to analyze the cases $p < 1/2$ and $p \geq 1/2$ separately. The average reputation between gossiping cooperators will always be positive.

After these first two rounds, the sign of the reputations versus cooperators and defectors might change however. From this point on, all gossiping cooperators will have a positive reputation amongst each other, so $\alpha_{ij}(m) = 1$ for $i, j \in N_3$ and $m \geq 2$. Then the expected change in reputation for $m > 2$ is given by

$$\langle \Delta R_{ij}(m)\rangle = \begin{cases} (1-\lambda)(2\alpha_{ij}(m)-1)+\lambda & \text{if } j \in N_1 \\ (\lambda-1)(2\alpha_{ij}(m)-1)-\lambda & \text{if } j \in N_2 \\ 1 & \text{if } j \in N_3 \end{cases} \tag{22}$$

Hence, if $\alpha_{ij}(m) = 1$ then $\langle \Delta R_{ij}(m)\rangle > 0$ for cooperators, and $\langle \Delta R_{ij}(m)\rangle < 0$ for defectors. That is, once they cooperate with cooperators, they will continue to cooperate, while for defectors, if they cooperate, after some time they will start defecting. Now suppose $\alpha_{ij}(m) = 0$. Then $\langle \Delta R_{ij}(m)\rangle > 0$ only if $\lambda > 1/2$ and $\langle \Delta R_{ij}(m)\rangle < 0$ only for $\lambda < 1/2$ against cooperators. For defectors, $\langle \Delta R_{ij}(m)\rangle > 0$ only if $\lambda < 1/2$ and $\langle \Delta R_{ij}(m)\rangle < 0$ only if $\lambda > 1/2$. This is summarized in Table II.

This suggests we can distinguish four different regimes of behavior, based on whether p and λ are larger or smaller than $1/2$, which we will briefly describe, and are summarized in Table III. We will demonstrate how to derive explicitly the payoffs $P_l(m)$ for the first regime, as the payoffs for the other regimes can be similarly dealt with. Due to lack of space, we provide the normalized payoffs only in the Appendix.

A. *Individualistic prejudiced*

When $\lambda < 1/2$ and $p < 1/2$ the gossiping cooperators listen more to themselves than to others, and they start out with some prejudice towards others—that is, they are more likely to defect than to cooperate in the first round. In the second round, they will cooperate with defectors, and defect against cooperators. Thereafter, the gossiping cooperators will always defect against cooperators, but cycles of cooperation and defection against defectors emerge. This is due to the fact that for $\lambda < 1/2$, whenever the reputation is negative, it will become positive, and whenever it is positive it will become negative again.

Suppose they have cooperated once, then the question is how many times they defect before they start another cooperative round. The change is $\langle \Delta R_{ij}(m)\rangle = (1-\lambda)-\lambda = 1-2\lambda$ if they defect and $\langle \Delta R_{ij}(m)\rangle = -(1-\lambda)-\lambda = -1$ if they cooperate. Hence, the question is for what number of rounds r we have $r(1-2\lambda)-1 > 0$, which implies $r > 1/(1-2\lambda)$. Hence, we will have about $1/(1-2\lambda)$ defections for every single cooperation, after the first round. This observation agrees also with the fact that the Win-Stay-Lose-Shift strategy cooperates with unconditional defectors half of the time (which is the case for $\lambda = 0$). Also, for $\lambda \to 1/2$, the number of defections per cooperation goes to infinity, showing they tend to always defect when $\lambda \to 1/2$.

We can simplify the calculations of the payoff for these cycles, by considering the average behavior for these cycles. Since for each cooperation we have $1/(1-2\lambda)$ defections we can spread this amount of cooperation over the complete cycle, and obtain an average cooperation of $(1-2\lambda)/(2-2\lambda)$ per round.

Let us first derive then the payoff $P_1(m)$ for cooperators. They will cooperate every round with everyone, so pay a cost of c, however, they will receive a benefit from themselves, so receive b. From the defectors, they will never receive any benefit. The gossiping cooperators will cooperate in the first round with probability p, and hence, the cooperators receive pb on average from them. In the subsequent round (since we assume here that $\lambda < 1/2$ and $p < 1/2$), the gossiping cooperators will defect with unconditional cooperators, that is, $\langle R_{ij}(2)\rangle < 0$ for $i \in N_3$ and $j \in N_1$ from (21) in this regime. From (22) we can conclude indeed that $\langle R_{ij}(m)\rangle < 0$ for further rounds. Putting all this together, we arrive at

$$P_1(m) = x_1(mb - mc) + x_2(-mc) + x_3(pb - mc). \tag{23}$$

Now let use derive the payoff $P_2(m)$ for the defectors. They will always defect, so never pay a cost c. From the cooperators they will receive a benefit b, and nothing from themselves. From the gossiping cooperators they will receive b with probability p, so pb on average as well. As we concluded earlier, cycles of cooperation and defection will emerge for the gossiping cooperators, but we can assume that they will only cooperate $(1-\lambda)/(1-2\lambda)$ rounds on average. Putting all this together, we obtain

$$P_2(m) = x_1(mb) + x_3\left(pb + (m-1)b\frac{1-\lambda}{1-2\lambda}\right) \tag{24}$$

Observe that necessarily all costs that are being made for type 1 appear as benefits to types 2 and 3, and likewise so for all other possibilities. This means that for the payoff $P_3(m)$ of the gossiping cooperators, we already discussed most of it. We only need to consider what the gossiping cooperators are doing amongst each other. Since they will always end up cooperating after round $m = 2$, this payoff is easy. Putting all

TABLE III
FOUR DIFFERENT REGIMES

	$p < 1/2$	$p > 1/2$
$\lambda < 1/2$	**Individualistic prejudiced** • Defect vs. cooperators • Cycles of cooperation vs. defectors	**Individualistic trusting** • Cooperate vs. cooperators • Cycles of cooperation vs. defectors
$\lambda > 1/2$	**Social prejudiced** • Cooperate vs. cooperators (except second round) • Defect vs. defectors (except second round)	**Social trusting** • Cooperate vs. cooperators • Defect vs. defectors

this together then yields

$$P_3(m) = x_1(mb - pc) + x_2\left(-pc - (m-1)c\frac{1-\lambda}{1-2\lambda}\right) + x_3((m-1)c + pc - (m-1)b - pc) \quad (25)$$

The payoffs for the other regimes can be similarly derived.

B. Individualistic trusting

When $\lambda < 1/2$ and $p \geq 1/2$ the gossiping cooperators still pay more attention to themselves, but they start cooperation somewhat easier in the first round. In the second round, they will cooperate with cooperators, and defect against defectors. After that, they will continue to cooperate with cooperators, while again cycles of cooperation and defection will appear, just as in the individualistic prejudiced regime.

Hence, the only difference between the trusting and the prejudiced individualistic regimes is that the gossiping cooperators cooperate with the cooperators in the trusting regime, instead of defecting them, as in the prejudiced case. This implies the gossiping cooperators are expected to perform relatively worse, especially for low λ.

C. Social prejudice

When $\lambda > 1/2$ and $p < 1/2$ the gossiping cooperators start to pay more attention to their peers, but will start with some reticence in the first round. They will always end up cooperating with cooperators, and defecting against defectors. However, in the second round, the reputation for the cooperators is expected to be negative, hence the gossipers will defect that round, while they will cooperate with defectors. The behavior versus defectors can indeed be seen as the limit of $\lambda \to 1/2$, but the behavior versus cooperators radically changes, since they will now end up always cooperating.

D. Social trusting

When $\lambda > 1/2$ and $p \geq 1/2$ they pay again more attention to their peers, and are more willing to cooperate. They will simply always cooperate with cooperators and defect against defectors, starting from the first round. This can be seen as the limit of $\lambda \to 1/2$ from the individualistic case, both for the behavior versus cooperators and defectors.

E. Fixed points and evolutionary stability

The dynamics for the four different regimes are qualitatively very similar. Example phase portraits for all regimes are shown in Fig. 3. For some parameter range there is an unstable fixed point on the x_2x_3 edge (so where $x_1^* = 0$), consisting of only gossiping cooperators and defectors. There are never any inner fixed points. For $p = 1$ the whole x_1x_3 edge are fixed points. So for $p = 1$ selection is neutral between unconditional and gossiping cooperators (if no defectors are present). In that case, a population of gossiping cooperators can drift towards unconditional cooperators. However, an invading type with alternative $p < 1$ then has a selective advantage.

The unstable fixed point on the x_2x_3 edge for the different regimes are:

Individualistic prejudiced

$$x_3^* = c\frac{1-2\lambda}{b-c} + pc\frac{2-2\lambda}{(b-c)(m-1)} \quad (26)$$

Individualistic trusting

$$x_3^* = c\frac{1-2\lambda}{b-c} - pc\frac{2-2\lambda}{(b-c)(m-1)} \quad (27)$$

Social prejudiced

$$x_3^* = c\frac{p+1}{(b-c)(m-2)} \quad (28)$$

Social trusting

$$x_3^* = c\frac{p}{(b-c)(m-1)} \quad (29)$$

The gossiping cooperators are exactly evolutionary stable whenever this fixed point exists, i.e. when $x_3^* < 1$, showing a transcritical bifurcation when $P_3(m) = 0$ for $x_3^* = 1$. The transition from the individual prejudiced regime to the individual trusting regime is especially sudden. This is due to the fact that the gossiping cooperators defect unconditional cooperators in the individual prejudiced regime, while they suddenly start cooperating with them in the trusting regime. This suggests individualistic, prejudiced gossiping cooperators perform relatively well in a friendly environment, whereas more socially oriented gossiping cooperators would do better in a tougher environment.

V. CONCLUSION

In this article we introduced a model for studying indirect reciprocity by explicitly considering the gossiping mechanism to allow for local reputations. We have shown that fixed point undirected cooperation networks are socially balanced.

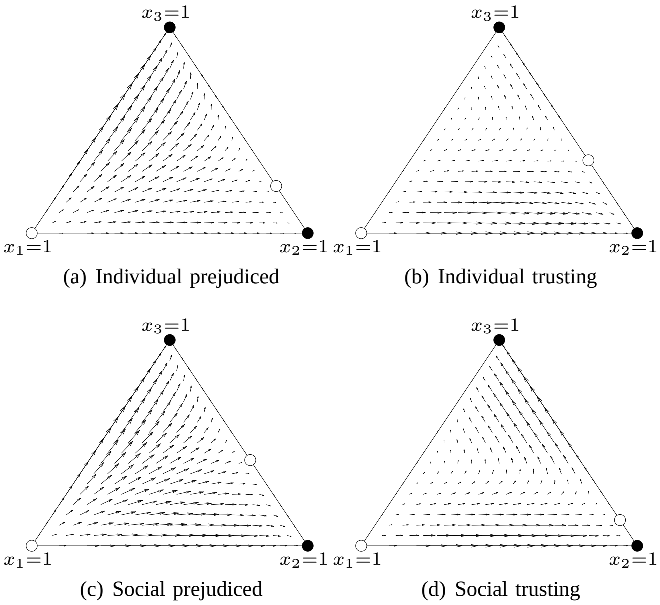

Fig. 3. Phase portrait of the social trusting regime. Parameters $b = 4$, $c = 1$, $m = 3$. For the prejudiced regime, $p = 1/4$ and for the trusting regime $p = 3/4$. The individual regime uses $\lambda = 1/4$, and the social regime $\lambda = 3/4$. Open circles signify unstable fixed points and closed (black) circles stable fixed points.

The model can be considered an (additional) argument why socially balanced situations should be more prevalent, apart from psychological theory [27]. It would also be interesting to compare this model to other dynamical models of social balance [23], [24], [28]. Furthermore, we have derived replicator equations to provide evolutionary stability conditions for the gossiping cooperators. This shows such a strategy could have a selective advantage, and possibly such a more socially oriented strategy could have evolved from the individual strategy.

The model as currently stated is far from realistic however. First of all, the interaction is assumed to be all-to-all, which clearly needs to be addressed. Combining this gossiping mechanism with for example active linking strategies [29], [30] or simply allowing some static interaction graph [31]–[33] should address this issue, and allow us to study indirect reciprocity on graphs. Secondly, gossip might be passed on further than only one step, thus allowing cascades of gossips, possibly reaching the entire population [34].

Even so, the model provides key insight in the working of gossip and reputation in the context of indirect reciprocity [3] and could provide some insight into how social norms are formed and upheld [12]–[14]. Yet the analysis of even this simple model is far from complete and needs to be studied further.

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REFERENCES

[1] M. A. Nowak, Evolutionary Dynamics: Exploring the Equations of Life, Belknap Press of Harvard University Press, 2006.

[2] M. A. Nowak, Five rules for the evolution of cooperation, Science 314 (5805) (2006) 1560–1563. doi:10.1126/science.1133755.

[3] M. A. Nowak, K. Sigmund, Evolution of indirect reciprocity, Nature 437 (7063) (2005) 1291–1298. doi:10.1038/nature04131.

[4] R. Alexander, The Biology of Moral Systems (Foundations of Human Behavior), Aldine Transaction, 1987.

[5] M. A. Nowak, K. Sigmund, Evolution of indirect reciprocity by image scoring, Nature 393 (6685) (1998) 573–577. doi:10.1038/31225.

[6] O. Leimar, P. Hammerstein, Evolution of cooperation through indirect reciprocity., Proceedings. Biological sciences / The Royal Society 268 (1468) (2001) 745–753. doi:10.1098/rspb.2000.1573.

[7] K. Panchanathan, A tale of two defectors: the importance of standing for evolution of indirect reciprocity, Journal of Theoretical Biology 224 (1) (2003) 115–126. doi:10.1016/S0022-5193(03)00154-1.

[8] H. Ohtsuki, Y. Iwasa, How should we define goodness? reputation dynamics in indirect reciprocity, Journal of Theoretical Biology 231 (1) (2004) 107–120. doi:10.1016/j.jtbi.2004.06.005.

[9] H. Ohtsuki, Y. Iwasa, The leading eight: Social norms that can maintain cooperation by indirect reciprocity, Journal of Theoretical Biology 239 (4) (2006) 435–444. doi:10.1016/j.jtbi.2005.08.008.

[10] H. Brandt, K. Sigmund, The logic of reprobation: assessment and action rules for indirect reciprocation, Journal of Theoretical Biology 231 (4) (2004) 475–486. doi:10.1016/j.jtbi.2004.06.032.

[11] M. Nakamaru, M. Kawata, Evolution of rumours that discriminate lying defectors., Evolutionary Ecology Research 6 (2) (2004) 261–283.

[12] M. Kandori, Social norms and community enforcement, The Review of Economic Studies 59 (1) (1992) 63–80. doi:10.2307/2297925.

[13] N. Elias, J. L. Scotson, The Established and the Outsiders, Sage Publications, London, 1994.

[14] N. E. Friedkin, Norm formation in social influence networks, Social Networks 23 (3) (2001) 167–189. doi:10.1016/S0378-8733(01)00036-3.

[15] E. Fehr, U. Fischbacher, Social norms and human cooperation, Trends in Cognitive Sciences 8 (4) (2004) 185–190. doi:10.1016/j.tics.2004.02.007.

[16] D. Kraines, V. Kraines, Evolution of learning among pavlov strategies in a competitive environment with noise, Journal of Conflict Resolution 39 (3) (1995) 439–466.

[17] D. Kraines, V. Kraines, Learning to cooperate with pavlov an adaptive strategy for the iterated prisoner’s dilemma with noise, Theory and Decision 35 (1993) 107–150.

[18] C. Wedekind, M. Milinski, Human cooperation in the simultaneous and the alternating prisoner’s dilemma: Pavlov versus generous tit-for-tat, Proceedings of the National Academy of Sciences 93 (1996) 2686–2689.

[19] S. Wasserman, K. Faust, Social network analysis: Methods and applications, Cambridge University Press, 1994.

[20] F. Harary, On the notion of balance of a signed graph, Michigan Math J 2 (1953) 143–146.

[21] D. Cartwright, F. Harary, On the coloring of signed graphs, Elemente der Mathematik 23 (1968) 85–89.

[22] J. Leskovec, D. Huttenlocher, J. Kleinberg, Predicting positive and negative links in online social networks, Proceedings of the 19th international conference on World wide web (2010) 641–650.

[23] S. A. Marvel, S. H. Strogatz, J. M. Kleinberg, The energy landscape of social balance, arXiv 0906.2893v1.

[24] T. Antal, P. L. Krapivsky, S. Redner, Dynamics of social balance on networks, Phys Rev E 72 (3) (2005) 036121+. doi:10.1103/PhysRevE.72.036121.

[25] N. Hummon, P. Doreian, Some dynamics of social balance processes: bringing heider back into balance theory, Social Networks 25 (2003) 17–49.

[26] J. Hofbauer, K. Sigmund, Evolutionary Games and Population Dynamics, Cambridge University Press, 1998.

[27] F. Heider, The psychology of interpersonal relations, Psychology Press, 1982.

[28] F. Radicchi, D. Vilone, S. Yoon, H. M. Ortmanns, Social balance as a satisfiability problem of computer science, Phys Rev E 75 (2007) 026106.

[29] J. M. Pacheco, A. Traulsen, M. A. Nowak, Coevolution of strategy and structure in complex networks with dynamical linking., Physical Review Letters 97 (2006) 258103.

[30] S. V. Segbroeck, F. C. Santos, A. Nowe, J. M. Pacheco, T. Lenaerts, The evolution of prompt reaction to adverse ties, BMC Evolutionary Biology 8 (1) (2008) 287+. doi:10.1186/1471-2148-8-287.

[31] E. Lieberman, C. Hauert, M. A. Nowak, Evolutionary dynamics on graphs, Nature 433 (7023) (2005) 312–316.

[32] H. Ohtsuki, C. Hauert, E. Lieberman, M. A. Nowak, A simple rule for the evolution of cooperation on graphs and social networks, Nature 441 (7092) (2006) 502–505.

[33] H. Ohtsuki, M. A. Nowak, The replicator equation on graphs, Journal of Theoretical Biology 243 (1) (2006) 86–97. doi:10.1016/j.jtbi.2006.06.004.

[34] P. G. Lind, L. R. da Silva, H. J. Herrmann, Spreading gossip in social networks, Phys Rev E 76 (3) (2007) 036117+. doi:10.1103/PhysRevE.76.036117.

APPENDIX

The payoffs have been normalized by subtracting $P_2(m)$ from every $P_i(m)$, i.e. $\hat{P}_i(m) = P_i(m) - P_2(m)$, which does not affect the dynamics [26]. Using this, we obtain the following normalized payoffs for the four regimes.

Individual prejudiced

$$\hat{P}_1(m) = -mc - (m-1)bx_3\frac{1-2\lambda}{2-2\lambda}, \tag{30}$$

$$\hat{P}_3(m) = (m-1)\left[x_3(b-c) - (cx_2 + bx_3)\frac{1-2\lambda}{2-2\lambda}\right] - pc. \tag{31}$$

Individual trusting

$$\hat{P}_1(m) = (m-1)bx_3\frac{1}{2-2\lambda} - mc, \tag{32}$$

$$\hat{P}_3(m) = (m-1)bx_3\frac{1}{2-2\lambda} - (m-1)c(x_1+x_3) - (m-1)cx_2\frac{1-2\lambda}{2-2\lambda} - pc. \tag{33}$$

Social prejudiced

$$\hat{P}_1(m) = (m-3)bx_3 - mc \tag{34}$$

$$\hat{P}_3(m) = (m-2)bx_3 - (m-1)cx_3 - (m-2)cx_1 - cx_2 - pc \tag{35}$$

Social trusting

$$\hat{P}_1(m) = (m-1)bx_3 - mc \tag{36}$$

$$\hat{P}_3(m) = (m-1)bx_3 - (m-1)c(x_1+x_3) - pc; \tag{37}$$

Obviously $\hat{P}_2(m) = 0$ for all regimes. By setting $\hat{P}_1(m) = \hat{P}_3(m) = \hat{P}_2(m) = 0$ and solving we are looking for interior fixed points, which do not exist, because the solution always contradict the bounds for λ, p, b, c and m. The fixed point on the line where $x_1^* = 0$ can be found by demanding $\hat{P}_3 = 0$ and solving for x_3^* and using $x_2^* = 1 - x_3^*$.

Now to investigate the stability of the fixed points of the replicator equation (17), we need to take a look at the eigenvalues of the Jacobian. If we write $f_i(x) = x_i(\hat{P}_i(m) - \hat{\bar{P}}(m))$, with $x = (x_1, x_2, x_3)$, the derivatives can be written as

$$\frac{\partial f_i}{\partial x_j} = x_i\left(\frac{\partial \hat{P}_i(m)}{\partial x_j} - \frac{\partial \hat{\bar{P}}(m)}{\partial x_j}\right) \text{ for } j \neq i, \tag{38}$$

$$\frac{\partial f_i}{\partial x_i} = (\hat{P}_i(m) - \hat{\bar{P}}(m)) + x_i\left(\frac{\partial \hat{P}_i(m)}{\partial x_i} - \frac{\partial \hat{\bar{P}}(m)}{\partial x_i}\right) \tag{39}$$

For the fixed points $x_i^* = 1$, the Jacobian simplifies as follows. Only the diagonal and $\partial f_j/\partial x_i$ is possibly non-zero, since $x_j = 0$ for $i \neq j$. Hence, the eigenvalues are contained exactly on the diagonal. This gives for $x_3^* = 1$

$$\frac{\partial f_2}{\partial x_2} = (\hat{P}_2(m) - \hat{\bar{P}}(m)) + x_2\left(\frac{\partial \hat{P}_2}{\partial x_2} - \frac{\partial \hat{\bar{P}}}{\partial x_2}\right), \tag{40}$$

but since $x_2^* = 0$ and $\hat{P}_2(m) = 0$, this simplifies to $-\hat{\bar{P}}(m) = -\hat{P}_3(m)$. Furthermore, since then

$$\frac{\partial \hat{\bar{P}}(m)}{\partial x_3} = \frac{\partial \hat{P}_3(m)}{\partial x_3} + \hat{P}_3 \tag{41}$$

we also obtain

$$\frac{\partial f_3}{\partial x_3} = -\hat{P}_3(m). \tag{42}$$

Since $\hat{P}_1(m) - \hat{\bar{P}}(m) < 0$ for $x_3^* = 1$ for $0 < p < 1$, the only condition is that $\hat{P}_3(m) > 0$ for $x_3^* = 1$, which is the case when the fixed point on the x_2x_3 edge (with $x_1^* = 0$) exists. The transcritical bifurcation then takes place when the stability of the fixed point $x_3^* = 1$ changes from instable to stable, i.e. when $\hat{P}_3(m) = 0$ for $x_3^* = 1$.

The fixed point $x_2^* = 1$ unfortunately is non-hyperbolic, so that we cannot use the linearization to study the stability. The function $V(x) = x_2$ for $x = (x_1, x_2, x_3)$ obtains its maximum clearly at $x_2 = 1 = x_2^*$, and has derivative

$$\frac{\dot{V}(x)}{V(x)} = (\log \dot{V}(x)) = (\log \dot{x}_2) = \frac{\dot{x}_2}{x_2} = -\hat{\bar{P}}(m). \tag{43}$$

so is a Lyapunov function if $\hat{\bar{P}}(m) < 0$. Since there exists small $x_1, x_3 > 0$ such that $\hat{\bar{P}}(m) < 0$, the point $x_2^* = 1$ is always stable.

The instability of the fixed point on the x_2x_3 edge (where $x_1^* = 0$) can be shown by only looking at this particular line, hence a simple one dimensional system. Since $\frac{\partial \hat{P}_3(m)}{\partial x_3} > 0$ for the fixed points in the one-dimensional system it is always unstable, hence it is also unstable in the larger two-dimensional system.