

Community detection in networks with positive and negative links

V. A. Traag^{1,2,*} and Jeroen Bruggeman²

¹*Department of Mathematical Engineering, Université Catholique de Louvain, Bâtiment Euler, Avenue G. Lemaître, 4 B-1348 Louvain-la-neuve, Belgium*

²*Department of Sociology and Anthropology, Faculty of Social and Behavioural Sciences, University of Amsterdam, Oudezijds Achterburgwal 185, 1012 DK Amsterdam, The Netherlands*

(Received 23 November 2008; revised manuscript received 9 July 2009; published 21 September 2009)

Detecting communities in complex networks accurately is a prime challenge, preceding further analyses of network characteristics and dynamics. Until now, community detection took into account only positively valued links, while many actual networks also feature negative links. We extend an existing Potts model to incorporate negative links as well, resulting in a method similar to the clustering of signed graphs, as dealt with in social balance theory, but more general. To illustrate our method, we applied it to a network of international alliances and disputes. Using data from 1993-2001, it turns out that the world can be divided into six power blocs similar to Huntington's civilizations, with some notable exceptions.

DOI: [10.1103/PhysRevE.80.036115](https://doi.org/10.1103/PhysRevE.80.036115)

PACS number(s): 89.75.Hc, 89.65.-s

I. INTRODUCTION

Many complex phenomena can be represented as networks, and subsequently be analyzed fruitfully [1–3]. One of the first targets of network analysis is the detection of communities on the basis of the links, i.e., the possibly valued, or weighted, arcs or edges that connect the nodes. Intuitively, an assignment of nodes to communities should be such that links within communities are relatively dense and between communities relatively sparse. This means we should compare actual densities to expected densities of links within and between communities. Furthermore, since nodes, for example humans or proteins, can be members of different communities at the same time, e.g., organizations or protein complexes, respectively, the assignment should allow for the possibility that communities overlap.

In approaches to find appropriate community assignments, much progress has been made in recent years [4–6] by using a concept known as *modularity* [7]. While current modularity approaches take for granted that links are positively valued, representing bonds or attraction, scientists in numerous fields grapple with networks that also have *negative* links that represent repel, conflict, or opposition, for example in neural networks, semantic webs, genetic regulatory networks, and last but certainly not least, in social networks.

In this paper, we generalize an existing Potts model [8] for positive links to incorporate negative links as well. We will follow the intuition that the assignment of nodes related by negative links should be done the opposite way of positive links, with negative links sparse within and more dense between communities, generalizing an old idea from social balance theory [9]. Finally, we apply our approach to a network of conflicts and alliances between countries.

Recently, it was shown that modularity might miss small communities embedded in larger ones [10], and is less accurate if the actual communities are highly different in size [11]. Our method has two balancing parameters that address

this problem to some extent [12]. Yet community detection through modularity remains a global rather than a local approach.

II. PROBLEM STATEMENT

We consider a directed graph G with n nodes and m links, which can be easily generalized to weighted graphs. We denote the total number of positive links in G as m^+ and the number of negative links as m^- , hence $m = m^+ + m^-$. We define the entries of the adjacency matrix of G as follows: if a positive link is present from node i to node j , $A_{ij} = 1$, if a negative link is present, $A_{ij} = -1$, and $A_{ij} = 0$ otherwise. For a weighted graph the link values, or weights, are denoted by w_{ij} . We separate the negative and positive links by setting $A_{ij}^+ = A_{ij}$ if $A_{ij} > 0$ and zero otherwise, and $A_{ij}^- = -A_{ij}$ if $A_{ij} < 0$ and zero otherwise, so $A = A^+ - A^-$. The positive and negative in- and outdegrees of i are defined as

$${}^\pm k_i^{\text{out}} = \sum_j A_{ij}^\pm \quad {}^\pm k_i^{\text{in}} = \sum_j A_{ji}^\pm \quad (1)$$

Our challenge is to assign each node i to one of c communities $\sigma_i \in \{1, \dots, c\}$. A complete configuration of community assignments is denoted by $\{\sigma\}$, which assigns each node $i = 1, \dots, n$ to a community $\sigma_1, \dots, \sigma_n$.

III. SOCIAL BALANCE

The challenge of community detection in networks with positive and negative links was first addressed by social balance theory, which has its origins in cognitive dissonance theory [13] from the 1950s. This theory is based on the notion that if two people are positively related, their attitudes toward a third person should match. For example, if Harry and Mary are positively related as friends, and both of them are related to John, they should both be related to him either positively or negatively. In either case their triad is said to be socially balanced. If Harry has a positive relationship with John while Mary is negatively related to John or vice versa, their triad is socially unbalanced. If all triads in a network are

*Corresponding author; vincent.traag@uclouvain.be

balanced, the network is said to be balanced. This definition was later generalized to cycles, a triad (a cycle of length 3) being a special case.

The question whether a balanced network can be divided into separate parts arises naturally. The challenge is to define clusters of nodes such that there are only positive links within clusters and negative links are between clusters. It was proven [9] that if a connected network is balanced, it can be split into two opposing clusters (and vice versa).

However, there is an ambiguous case. If a triad has only negative relationships, it is neither balanced nor can it be split into two clusters. But it can obviously be split into three clusters. In order to accommodate for this possibility, the definition of balance was generalized to k -balance [14,15]. A network is k -balanced if it can be divided into k different clusters, each cluster having only positive links within itself, and negative links with other clusters. It can be proven [14] that a network is balanced if and only if it contains no cycles with exactly one negative link. The intuition is simple. Suppose there is a cycle $v_1 v_2 \dots v_k v_1$ with one negative link, say between v_1 and v_k , and only positive links between the remainder nodes, then 1 and k are both positively and negatively connected, and the cycle is unbalanced. But if in this cycle there is also a negative link between i and j , and $1 \leq i < j \leq k$, we can split the cycle in two parts, one cluster from 1 to i and one from j to k . If there are more than two negative links, we can split up the cycle analogously into more clusters.

IV. FRUSTRATION

In reality, however, social networks are rarely, if ever, fully k -balanced. The question then becomes whether we can still cluster nodes. Obviously, there are some links that make a network unbalanced. The number of such links can be expressed as an amount of *frustration*. Links that contribute to frustration are negative links within clusters and positive links between clusters. So the following expression should be minimized

$$\sum_{ij} \lambda A_{ij}^- \delta(\sigma_i, \sigma_j) + (1 - \lambda) A_{ij}^+ (1 - \delta(\sigma_i, \sigma_j)), \quad (2)$$

where $\delta(\sigma_i, \sigma_j) = 1$ if $\sigma_i = \sigma_j$ and zero otherwise, and λ is a parameter through which the contribution of the two types of links can be balanced. This is the approach taken in [16,17].

The objective, then, is to find a clustering $\{\sigma\}$ such that the frustration is minimized. Since the term $(1 - \lambda) A_{ij}^+$ does not depend on the specific configuration and is therefore irrelevant for finding the minimum, we can simplify the above expression to

$$\sum_{ij} (\lambda A_{ij}^- - (1 - \lambda) A_{ij}^+) \delta(\sigma_i, \sigma_j), \quad (3)$$

We can now see that only for $\lambda = 1/2$ we retrieve $A = A^+ - A^-$, up to a multiplicative constant of 2. Using any other value for λ would change the minimum found, but changing λ is the same as altering the (weights of the) original network. Setting $\lambda = 1/2$ accordingly, we can simplify further, and now define frustration as

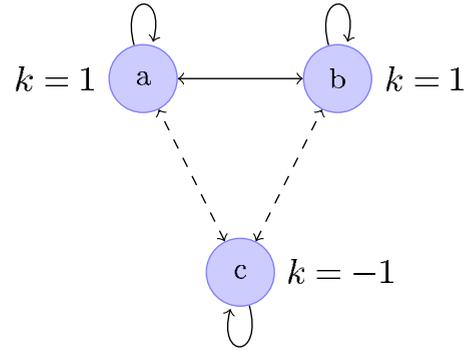


FIG. 1. (Color online) Network illustrating the shortcoming of modularity when negative links are present. The dashed links have a weight of -1 and the others have a weight of 1 . Using expected link values $p_{ij} = k_i^{out} k_j^{in} / m$, the expected values equal the actual values, $p_{ij} = A_{ij}$. Hence the modularity $Q = 0$ for all configurations.

$$F(\{\sigma\}) = - \sum_{ij} A_{ij} \delta(\sigma_i, \sigma_j). \quad (4)$$

However, frustration does not generalize to a network with only positive links. In that case, frustration groups together all nodes into one cluster. Even if there are some negative links, frustration will cluster together very sparsely connected nodes. It's therefore clear that this approach does not match with current methods of community detection. Preferably, there should not be a distinction between methods for positive and others for negative links, but there should be one method for both.

V. MODULARITY

In approaches to find appropriate community assignments in networks with only positive links, much progress has been made recently [4–6]. The principal method for detecting communities is through modularity optimization, which boils down to clustering nodes based on the link densities within and between communities. The link densities should be high within communities and low between communities.

The ordinary¹ definition of modularity for directed graphs [19] is

$$Q(\{\sigma\}) = \frac{1}{m} \sum_s m_{ss} - [m_{ss}], \quad (5)$$

where m_{ss} is the actual number of links within a community s , $\sum A_{ij} \delta(\sigma_i, s) \delta(\sigma_j, s)$, and $[m_{ss}]$ is the expected number of such links, $\sum p_{ij} \delta(\sigma_i, s) \delta(\sigma_j, s)$, where p_{ij} is some expected value in a random null model. The expected values are constrained by $\sum p_{ij} = m$, because the random null model should have the same number of links as the actual network. Taking degrees into account, a sensible expectation is $p_{ij} = k_i^{out} k_j^{in} / m$, which was used in the original definition [19].¹

¹Keep in mind that we consider directed graphs. Therefore the sum of all degrees is m , not $2m$, which is reflected in some minor changes to the original definitions [18]; see also [19].

A straightforward generalization to weighted graphs is to set $A_{ij}=w_{ij}$ and to take the degree measure as the sum of the link weights. For graphs with negative weights, however, a problem arises, illustrated in Fig. 1. The weighted degree of the three nodes a , b and c is $k_a=1$, $k_b=1$ and $k_c=-1$. The total weighted degree is $m=\sum w_{ij}=1$. The expected values $p_{ij}=k_i^{\text{out}}k_j^{\text{in}}/m$ equal the edge weights w_{ij} . Hence $A_{ij}-p_{ij}=0$ for all links, and each possible community configuration results in a modularity $Q=0$, while the appropriate configuration is clear from the figure: a and b belong to the same community, and c to another community [20]. Some adaptations of modularity are therefore required to detect communities in networks with (also) negative links.

VI. POTTS MODEL AND EXTENSION

To get rid of the shortcomings of frustration and modularity, we will attempt to modify the latter. After all, for many networks with only positive links, modularity has shown to uncover valid community structures in multiple areas [21]. We will extend the original definition of modularity to allow for negative links, by phrasing our challenge as a Potts model [22]. This model will also show up a close connection between social balance and modularity.

We start out by treating the positive and the negative links separately. Mimicking the approach taken by Reichardt and Bornholdt [8] we first define a Hamiltonian for the positive part, which represents the “energy” of a given configuration $\{\sigma\}$. We reward internal positive links by a_{ij} and penalize absent internal positive links by b_{ij} , which leads to

$$\mathcal{H}^+(\{\sigma\}) = \sum_{ij} [-a_{ij}A_{ij}^+ + b_{ij}(1 - A_{ij}^+)]\delta(\sigma_i, \sigma_j). \quad (6)$$

Setting $a_{ij}=1-b_{ij}$ and $b_{ij}=\gamma^+p_{ij}^+$, where p_{ij}^+ represents the expected (positive) link between i and j , allows us to simplify the above equation to²

$$\mathcal{H}^+(\{\sigma\}) = - \sum_{ij} (A_{ij}^+ - \gamma^+p_{ij}^+)\delta(\sigma_i, \sigma_j). \quad (7)$$

which is the Potts model analyzed by Reichardt and Bornholdt [8] if only positive links are present. We define the negative part analogously, but now we penalize internal negative links and reward absent negative internal links,

$$\mathcal{H}^-(\{\sigma\}) = \sum_{ij} (A_{ij}^- - \gamma^-p_{ij}^-)\delta(\sigma_i, \sigma_j). \quad (8)$$

The effect of the negative links on the energy of the entire configuration is opposite to the effect of the positive links. Combining the two Hamiltonians into one yields

$$\mathcal{H}(\{\sigma\}) = (1 - \lambda)\mathcal{H}^+(\{\sigma\}) + \lambda\mathcal{H}^-(\{\sigma\}), \quad (9)$$

where λ plays a similar role as in frustration, of balancing the effects of positive and negative links. As explained earlier, it makes sense to weigh the contributions of each part equally, thus $\lambda=1/2$. To illustrate, let us define a new matrix $B_{ij}=(1-\lambda)A_{ij}^+-\lambda A_{ij}^-$, and construct the Hamiltonian for this

altered network by setting $\lambda=1/2$. Since the expected values for B are $p_{ij}^+=(1-\lambda)p_{ij}^+$ and $p_{ij}^-=\lambda p_{ij}^-$, the Hamiltonian for B is equivalent to the one for A , up to a multiplicative constant of 2. So we may indeed set $\lambda=1/2$ and then simplify the above Hamiltonian (up to the multiplicative constant of 2) to

$$\mathcal{H}(\{\sigma\}) = - \sum_{ij} [A_{ij} - (\gamma^+p_{ij}^+ - \gamma^-p_{ij}^-)]\delta(\sigma_i, \sigma_j), \quad (10)$$

which is the measure that we optimize to detect a community structure in networks with both positive and negative links. It can be easily seen that when the network is positive (and $\gamma^\pm=1$) we obtain

$$Q(\{\sigma\}) = - \frac{1}{m}\mathcal{H}(\{\sigma\}). \quad (11)$$

So minimizing the Hamiltonian (10) is the same as maximizing modularity. In fact we just compare the original network to the appropriate negative link null model, which wasn't the case in the original modularity [18] and in the Potts model [8].

The simplest version of the expected values, p_{ij}^\pm , is obtained by just regarding the proportion of positive or of negative links in the network, $p_{ij}^\pm=m^\pm/n(n-1)$. If we want to take the degree distribution into account, then $p_{ij}^\pm=k_i^{\text{out}}k_j^{\text{in}}/m^\pm$. The modularity given in [20] also defines this negative link null model appropriately, and is a special case of ours.

When $\gamma^+=\gamma^-=0$, the Hamiltonian (10) equals the frustration (4) of the network, and if the network is also balanced and complete (no missing edges), minimizing the Hamiltonian (10) yields the same result as minimizing the frustration (4). This can be pointed out by defining the probabilities by $p_{ij}^\pm=m^\pm/n(n-1)$, and by allowing the complete and balanced network to consist only of link values $A_{ij}\in\{-1, 1\}$. Then, as long as $\gamma^+m^+-\gamma^-m^-<n(n-1)$, the coupling $A_{ij}-(\gamma^+p_{ij}^+-\gamma^-p_{ij}^-)$ between each positively associated pair of nodes is positive. Hence, the configuration produced by minimizing the Hamiltonian is the same as when minimizing the frustration.

VII. THE GROUND STATE

Finding the actual minimum of the Hamiltonian—the so-called ground state—is NP hard [23], and therefore only heuristic methods can be applied. Our modularity (11) can be easily integrated with existing algorithms for modularity optimization, such as eigenvector [6], extremal optimization [4], fast unfolding [24], or simulated annealing [8,25]. We opted for simulated annealing [26] to minimize the Hamiltonian (10) because it performs well in standard performance tests [8,27–29], although it's not the fastest algorithm [24,30]. Here we will give a short overview of how to adapt the simulated annealing approach [8] to incorporate negative links.

First, it's convenient to define the adhesion between community r and s , similar to [8],

²Notice that if we have a weighted network, $a_{ij}=w_{ij}-b_{ij}$.

$$a_{rs} = (m_{rs}^+ - m_{rs}^-) - ([m_{rs}^+] - [m_{rs}^-]),$$

where $m_{rs}^+ = \sum A_{ij}^+ \delta(\sigma_i, r) \delta(\sigma_j, s)$ is the actual number of arcs from r to s and $[m_{rs}^+] = \sum \gamma^+ p_{ij}^+ \delta(\sigma_i, r) \delta(\sigma_j, s)$ is the expected number of arcs from r to s . Hamiltonian (10) can be rewritten accordingly,

$$\mathcal{H}(\{\sigma\}) = \sum_s a_{ss} = - \sum_{r \neq s} a_{rs}, \quad (12)$$

where r and s are communities in $\{\sigma\}$.

In order to minimize the Hamiltonian, we consider the effect of moving a single node from one community to another, expressed in terms of adhesion. More specifically, moving node v from r to s results in the following change,

$$\Delta \mathcal{H}(\sigma_v: r \rightarrow s) = (a_{vr} + a_{rv}) - (a_{vs} + a_{sv}), \quad (13)$$

where a_{vr} is the adhesion between node v and its complement in community r . Let us write the mutual adhesion of a node v and a community r as $\alpha_v(r) = a_{vr} + a_{rv}$. If, for the sake of argument, the mutual adhesion of v and s is larger than the mutual adhesion of v and r , $\Delta \mathcal{H}(\sigma_v: r \rightarrow s) = \alpha_v(r) - \alpha_v(s)$ decreases the Hamiltonian. In other words, v has more positive, or less negative, links than expected to s than to r , and moving v to s would improve the configuration.

To each move we can assign a probability [26]

$$\Pr(\sigma_v: r \rightarrow s) = \frac{\exp(\beta \Delta \mathcal{H}(\sigma_v: r \rightarrow s))}{\sum_i \exp(\beta \Delta \mathcal{H}(\sigma_v: r \rightarrow i))},$$

where T is the temperature and $\beta = 1/T$. By slowly decreasing the temperature, the probability of moving to another state approaches the maximum possible, thereby forcing the system into its minimum energy, i.e., the ground state. Notice that in principle the probabilities are dependent on the total energy $\mathcal{H}(\sigma_v: r \rightarrow s)$, but since $\mathcal{H}(\sigma_v: r \rightarrow s) = \mathcal{H}(\{\sigma\}) + \Delta \mathcal{H}(\sigma_v: r \rightarrow s)$ we can simplify to the equation stated above.

The algorithm iterates randomly over the nodes a number of times, after which the temperature is decreased to a lower temperature T' stepwise, and usually (although not necessarily) $T' = 0.99T$. The iterations and the lowering of the temperature are continued until there are no further (significant) improvements. Any further changes would result in a higher energy, which we do not want, hence the resulting configuration of minimum energy is our solution $\{\sigma\}$. Herein, for any set of nodes u , its mutual adhesion to its own community s is stronger than to any other community r , $\alpha_u(s) \geq \alpha_u(r)$, which is clear when one looks at Eq. (13).

Furthermore, the cohesion, or self-adhesion, a_{ss} of a community is always positive, $a_{ss} \geq 0$, and the mutual adhesion between two communities is always negative, $a_{rs} + a_{sr} \leq 0$. If the cohesion were negative, we could then move a set of nodes to another community and thereby decrease the energy, which would contradict the fact that the system is in the ground state.

In fact these last two inequalities can be rephrased, which yields some insight into the effect of the parameters γ^\pm . If we assume, for analytic purposes, that $p_{ij}^\pm = p^\pm$

$= m^\pm / n(n-1)$, the expected values become $[m_{rs}^\pm] = p^\pm n_r n_s$ for $r \neq s$ and $[m_{ss}^\pm] = p^\pm n_s (n_s - 1)$, where n_s is the number of nodes in community s . Writing this out we arrive at

$$\frac{m_{ss}^+ - m_{ss}^-}{n_s(n_s - 1)} \geq \frac{\gamma^+ m^+ - \gamma^- m^-}{n(n-1)} \geq \frac{(m_{rs}^+ - m_{rs}^-) + (m_{sr}^+ - m_{sr}^-)}{2n_r n_s}, \quad (14)$$

wherein the middle term is a sort of global density. Hence by changing γ^\pm we change the threshold for clustering nodes together versus keeping them apart. Either way, the density within a community is always higher than the global density of the network, while the density between communities is always lower than the global density. Increasing γ^+ raises the threshold for nodes to be clustered, and will (generally) result in smaller communities detected, possibly embedded in larger and sparser communities. Increasing γ^- has the opposite effect and lowers the threshold, and will (generally) result in a configuration of larger communities.

VIII. APPLICATION

To show how our method can be applied to an empirical network, we analyze international relations taken from the Correlates of War [31,32] data set over the period 1993–2001, where military alliances can be represented by positive links and disputes by negative links. The data set contains a wide variety of disputes, for example border tensions between Colombia and Venezuela, the deployment of Chinese submarines to Japanese islands, and Turkish groups entering Iraqi territory. Disputes were assigned hostility levels, from “no militarized action” to “interstate war,” and we chose the mean level of hostility between two countries over the given time interval as the weight of their negative link. The alliances we coded one of three values, for (1) entente, (2) non-aggression pact, or (3) defense pact. The disputes w_{ij}^- and alliances w_{ij}^+ are both normalized to values in the interval $w_{ij}^\pm \in [0, 1]$ which then bear equal weight in the overall link value $w_{ij} = w_{ij}^+ - w_{ij}^-$. The largest connected component consists of 161 nodes (countries) and 2517 links (conflicts and alliances).

The result of the analysis ($Q=0.561$) is shown in Fig. 2. Countries of the same color (or pattern) belong to the same community, which in this context is more appropriately labeled a *power bloc*. How strongly a country belongs to its power bloc can be determined by the adhesion $\alpha_v(s)$ a node has to its community. The power blocs can be identified as follows: (1) the West; (2) Latin America; (3) Muslim World; (4) Asia; (5) West Africa; and, (6) Central Africa. If we detect communities by using only positive links, there is an agreement of about 64% with the configuration in Fig. 2, while if using only negative links, there is an agreement of about 30%.

Our result resembles the configuration depicted in Huntington’s renowned book *The Clash of Civilizations* [33], with a few notable exceptions. The West African power bloc is an additional insight that is absent in Huntington’s configuration. A major difference with Huntington is that China

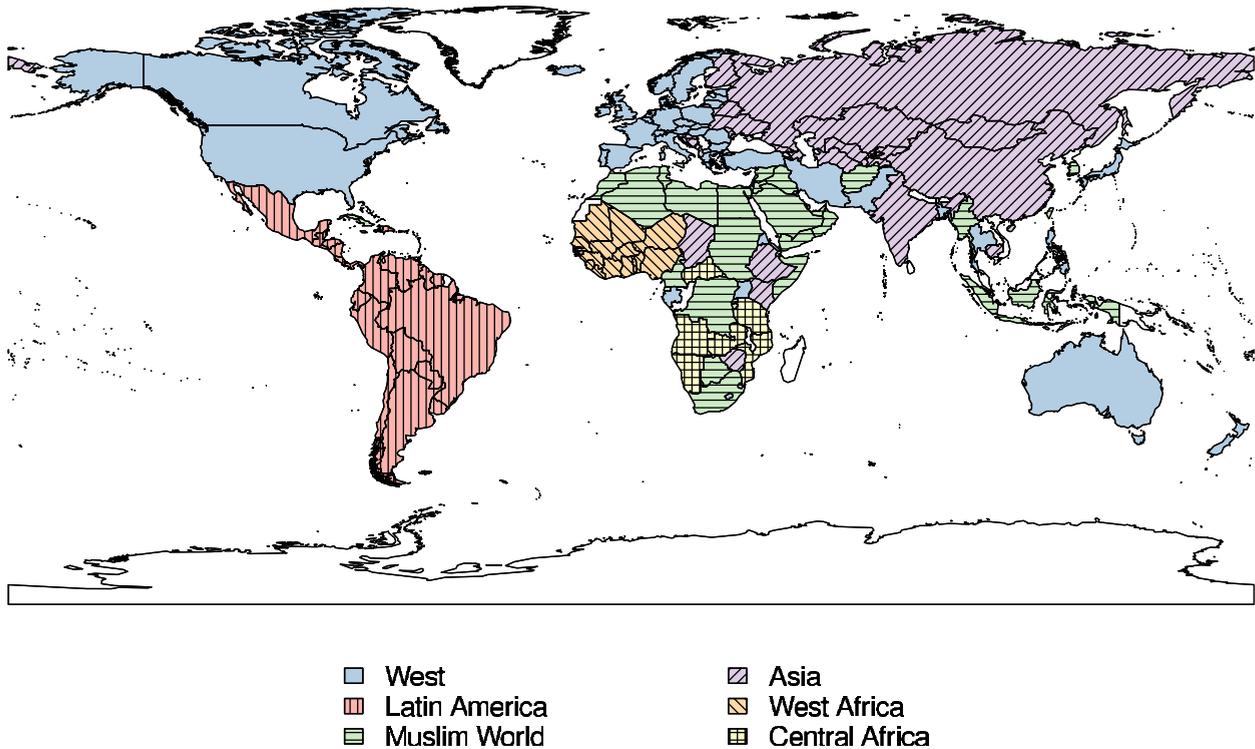


FIG. 2. (Color online) Map of the communities in the conflict and alliance network found using the algorithm described in the text ($Q=0.561$, $\gamma^{\pm}=1$).

itself does not constitute a separate bloc, nor does Japan or India. Some other noteworthy differences are Pakistan and Iran which are grouped with the West, while South Korea and South Africa are grouped with the Muslim World.

If we run the algorithm with $\gamma^{\pm}=0.1$ and $\gamma^{-}=1$, North America merges with Latin America, while Europe becomes an independent community, and North Africa and the Middle East align with Russia and China. When setting $\gamma^{\pm}=1$ and $\gamma^{-}=2$, in contrast, former Soviet countries separate from Russia and form an independent community. Using a range of values for γ^{\pm} , one can detect various layers in the community structure.

Our configuration does not imply that conflicts take place between power blocs only, as 24% of all conflicts actually take place *within* blocs. For example, Georgia and Russia had serious conflicts, and DR of Congo and Rwanda had theirs, but each of these pairs is grouped together nevertheless. In these cases, the alliances overcame the conflicts in the grouping, confirming that a configuration of international relations is more than the sum of bilateral links.

Our political analysis here is limited, since we wish to demonstrate the method rather than present a complete coverage of international alliances and disputes. Other approaches that could be brought into play are the democratic peace theory [34,35], which predicts few conflicts between democratic countries but fails to predict that in actuality, most conflicts occur between democratic and nondemocratic countries; the realist school [36], which emphasizes geopolitical concerns; and, the trade-conflict theory [37], which argues that (strong) trade relations diminish the probability of a dispute, or lower its intensity. In sum, although Hunting-

ton's configuration of civilizations was questioned [38,39], it seems to be fairly robust and with some marked exceptions is confirmed by our analysis.

IX. CONCLUSION

We have extended the existing Potts model by adapting the concept of modularity to detect communities in complex networks where both positive and negative links are present. This approach solves a long-standing problem in the theory of social balance, namely the clustering of signed graphs.

As a case in point, we have analyzed a social network of international disputes and alliances. Other applications could be networks of references on the Web [40] or in blogs [41]. If in these data positive and negative references are distinguished, our method makes possible to detect not only thematic clusters, but also positional clusters with internal agreement and external disagreement.

For network data, the model's parameters (γ^{\pm}) can be used to find smaller (sub) communities, although there is currently no theoretical guidance to choose parameter values [12]. Even if there were such guidance, the modularity approach intrinsically aims at global rather than local optimization. Our implementation is based on simulated annealing [8,26], which performs quite well with standard tests, although for very large networks, faster algorithms will be necessary [24].

Whatever algorithms future researchers will use, or improvements of the concept of modularity they will develop, being able to detect communities in networks with both positive and negative links is important in numerous fields of

science, and a stepping stone toward further analyses of complex networks.

ACKNOWLEDGMENTS

The authors like to thank anonymous referees and Jean-Charles Delvenne for their valuable comments and advice.

VT acknowledges support from a grant “Actions de recherche concerts Large Graphs and Networks” of the “Communauté Française de Belgique” and from the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office.

-
- [1] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D. U. Hwang, *Phys. Rep.* **424**, 175 (2006).
- [2] S. Dorogovtsev, A. Goltsev, and J. Mendes, *Rev. Mod. Phys.* **80**, 1275 (2008).
- [3] J. Bruggeman, *Social Networks: An Introduction* (Routledge, London, 2008).
- [4] J. Duch and A. Arenas, *Phys. Rev. E* **72**, 027104 (2005).
- [5] G. Palla, I. Derényi, I. Farkas, and T. Vicsek, *Nature (London)* **435**, 814 (2005).
- [6] M. E. J. Newman, *Phys. Rev. E* **74**, 036104 (2006).
- [7] M. Girvan and M. E. J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 7821 (2002).
- [8] J. Reichardt and S. Bornholdt, *Phys. Rev. E* **74**, 016110 (2006).
- [9] F. Harary, *Mich. Math. J.* **2**, 143 (1953).
- [10] S. Fortunato and M. Barthélemy, *Proc. Natl. Acad. Sci. U.S.A.* **104**, 36 (2007).
- [11] H. Du, D. R. White, Y. Ren, and S. Li (to be published).
- [12] J. M. Kumpula, J. Saramäki, K. Kaski, and J. Kertész, *Eur. Phys. J. B* **56**, 41 (2007).
- [13] F. Heider, *J. Psychol.* **21**, 107 (1946).
- [14] J. A. Davis, *Hum. Relat.* **20**, 181 (1967).
- [15] D. Cartwright and F. Harary, *Elemente der Mathematik* **23**, 85 (1968).
- [16] P. Doreian and A. Mrvar, *Soc. Networks* **18**, 149 (1996).
- [17] P. Jensen, *Phys. Rev. E* **74**, 035101 (2006).
- [18] M. E. J. Newman and M. Girvan, *Phys. Rev. E* **69**, 026113 (2004).
- [19] E. A. Leicht and M. E. J. Newman, *Phys. Rev. Lett.* **100**, 118703 (2008).
- [20] S. Gomez, P. Jensen, and A. Arenas, *Phys. Rev. E* **80**, 016114 (2009).
- [21] J. Reichardt, *Structure in Complex Networks*, Lecture Notes in Physics (Springer, New York, 2008).
- [22] F. Y. Wu, *Rev. Mod. Phys.* **54**, 235 (1982).
- [23] U. Brandes, D. Delling, M. Gaertler, R. Görke, M. Hofer, Z. Nikoloski, and D. Wagner, e-print arXiv:physics/0608255v2.
- [24] V. D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre, *J. Stat. Mech.: Theory Exp.* **2008**, P10008 (2008).
- [25] R. Guimerà, S. Mossa, A. Turttschi, and L. A. Amaral, *Proc. Natl. Acad. Sci. U.S.A.* **102**, 7794 (2005).
- [26] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi, *Science* **220**, 671 (1983).
- [27] L. Danon, D. Aguilera, J. Duch, and A. Arenas, *J. Stat. Mech.: Theory Exp.* (2005), P09008.
- [28] R. Guimerà and N. L. A. Amaral, *Nature (London)* **433**, 895 (2005).
- [29] A. Lancichinetti, S. Fortunato, and F. Radicchi, *Phys. Rev. E* **78**, 046110 (2008).
- [30] M. E. J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **103**, 8577 (2006).
- [31] F. Ghosn, G. Palmer, and S. Bremer, *Conflict Management and Peace Science* **21**, 133 (2004).
- [32] D. M. Gibler and M. Sarkees, *J. Peace Res.* **41**, 211 (2004).
- [33] S. P. Huntington, *The Clash of Civilizations and the Remaking of World Order* (Simon & Schuster, New York, 1996).
- [34] P. R. Hensel, G. Goertz, and P. F. Diehl, *J. Polit.* **62**, 1173 (2000).
- [35] D. Tocqueville and A., *Democracy in America* (Perennial, New York, 2000).
- [36] H. Kissinger, *Diplomacy* (Simon & Schuster, New York, 1994).
- [37] S. W. Polachek, J. Robst, and Y. C. Chang, *J. Peace Res.* **36**, 405 (1999).
- [38] E. A. Henderson and R. Tucker, *Int. Stud. Q.* **45**, 317 (2001).
- [39] B. M. Russett, J. R. Oneal, and M. Cox, *J. Peace Res.* **37**, 583 (2000).
- [40] G. W. Flake, S. Lawrence, C. L. Giles, and F. M. Coetzee, *ptComputer* **35**, 66 (2002).
- [41] L. A. Adamic and N. Glance, *LinkKDD '05: Proceedings of the 3rd International Workshop on Link Discovery* (ACM Press, New York, NY, 2005), pp. 36–43.